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Implementation of Didactic Engineering for the Understanding of Mathematical Concepts

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Abstract

Throughout history, there have been many mathematical concepts that have generated difficulties in students, possibly due to their abstract nature, which demands a series of mental skills that must be strengthened throughout the formative process as the brain matures, as has been reviewed in a good deal of research in the field of mathematics education. The concepts of differential calculus do not escape this complex panorama, especially in the derivative area. Several works have highlighted that teachers' promotion of algorithmic teaching causes the main weakness in understanding the concept of the derivative in students. In this context, didactic engineering was implemented to guide the process of teaching the derivative to propose didactic situations that promote its true understanding. Using a knowledge test, it was intended to identify the degree of understanding acquired by a group of teachers in training in mathematics at a public university in the Colombian northeast, to design pedagogical processes of intervention that guarantee their true learning

Keywords: Concept of the derivative, graphic register, didactic engineering, semiotic registers of representation.

1. Introduction

The Differential Calculus subject marks the beginning of mathematics in higher education and is built from the concept of function that acts as a connector between the two levels of schooling (Prada et al., 2017; Prada-Núñez et al., 2016). Now, in Hernández-Suárez et al. (2017), it is highlighted that difficulties have been identified in the understanding of various concepts related to university mathematics, especially in the subject of Differential Calculus, caused by a poor orientation on the part of teaching that indirectly promotes the

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mastery of algebraic processes, restricting understanding and thus leading to a poor formation in learning and appropriation of the subjects studied (García, 2013; Pinto & Parraguez, 2015; Gallo et al., 2017). In addition, tall (1993) states that there are deficiencies in the learning of calculus in terms of its visual representation, with a preference for numerical and symbolic manipulation, without relating it to problem-solving.

In this perspective, Hitt (2003) sustains the need for mathematical visualization connecting with different representations in the learning of calculus since by performing merely algebraic manipulations, understanding is inefficiently addressed, limiting the meaning of the mathematical notions that are sought to be developed, which coincides with what is affirmed in the research of Trujillo et al. (2019) and Pineda et al. (2019). The above is related to Duval (2006), who states that

> ...only in mathematics where a wide and complex set of transformations of representations is required for thinking and because a dualistic approach to mathematical activity leads to denying its cognitive importance. Conceptual understanding arises from coordinating the various semiotic systems used and realizing that the specific representation for each semiotic system is a cognitive condition for comprehension (p. 166).

Within this order of ideas, the derivative is a topic that is part of the micro curriculum of the subject of differential calculus, which is not alien to the above premise, presenting difficulties in the understanding of the concepts related to it, being an indicator susceptible to research that has led to the development of different works focused on identifying and enhancing a better understanding of the derivative of a function (Gómez, 2020a; Gómez, 2020b; Ramírez et al., 2020). For example, park (2013), after guiding a unit in training in this concept, identified the conceptions presented by the students, determining that they were not entirely constructed, building different ideas of the derivative of a general and particular function (having better assimilation in this), without relating the notions of derivative and slope of the tangent line as the differential in the abscissa tends to zero.

In Ruiz et al. (2018), the objective was intended to visualize scenarios of concepts attached to the derivative by the students, for which a teaching methodology was taught through the use of mobile devices, obtaining significant progress in terms of the conceptualization of the ratio of change, also warning that the assimilation of the concept of derivative as the operative resolution of exercises without relating them to their respective geometric understanding using the notion of limit takes precedence; which is similar to the evidence obtained by Briceño et al. (2018) where after applying a geometric-variational type instrument related to the concept of the derivative and focused on determining the existing ideas on the part of the students, tendencies were obtained towards an algebraic assimilation related to the limit formula and the rules of derivation, disregarding their respective graphic understanding, also having non-complementary notions of the tangent; as a straight line that only touches the function at a single point. These statements turn out to be coincident with some conclusions coming from works previously carried out by Vrancken et al. (2010), Rodríguez-Pérez (2015), Gutiérrez et al. (2017) and Saraza & Prada-Núñez (2017). The work of Rojas & Del Rosario (2020) highlights the importance of problem-solving as the ultimate goal of the teaching process of mathematical concepts or what is stated by Duarte et al. (2018), who recognize the importance of feelings and emotions in the processes of teaching and learning mathematics.

Based on the above, it is necessary to bring into consideration the importance of promoting a good understanding of this subject from a graphical perspective, as stated by Weigand (2014), who argues that it is essential to promote a good understanding of the concept of the derivative that transcends the syntactic rules of derivation and thus creates solid foundations for the continuation of the line of calculus and its articulation with other disciplines. This premise heads the purpose of this research work, which adopts the didactic engineering methodology to promote an understanding of the concept of the derivative from a graphical perspective without neglecting its corresponding algebraic assimilation.

2. Method

The research methodology implemented in this work corresponds to the so-called didactic engineering, according to Calderón and León (2012), which presents a correlation between an epistemic analysis and the implementation of a didactic design. The authors state that the essential particularity of this research method is the analysis given during the design of the activities (a priori) and the analysis after their respective implementation (a posteriori). Artigue et al. (1995) characterize it as a method of didactic realizations that involves the construction and execution of teaching sequences in order to subsequently analyze the results found, establishing a series of phases to be followed in the respective process: a) Phase 1: Preliminary analysis; b) Phase 2: Conception and a priori analysis of engineering didactic situations; c) Phase 3: Experimentation; and, d) Phase 4: A posteriori analysis and evaluation.

In this sense, the present research is aimed at exhausting each of the phases in the didactic engineering methodology, presenting in this document the results of each one of them, where an epistemic analysis of the evolutionary development of the concept of derivative was carried out first. Secondly, an a priori analysis, an a priori analysis, based on a documentary-type tracing derived from state of the art around the understanding of the concept of derivative, focused on the identification of the contributions previously made by other researchers in terms of the shortcomings or potentialities that are continuously presented at the time of orienting this subject. Subsequently, based on the epistemological analysis and the a priori analysis, a questionnaire was structured and applied, which allowed identifying the learning difficulties associated with the concept of the derivative. Then, a didactic sequence was designed and applied, based on the articulation of different semiotic registers of representation around the concept under study, to end with the application of the same knowledge test to identify the progress made by the participants.

The population subject to this work were 40 students enrolled in the first semester taking the Differential Calculus course in the academic program of Bachelor's Degree in Mathematics at a public university in the northeastern part of Colombia. In addition, a non-probabilistic sample of thirteen students was selected from them who wished to participate in a counseling program that demanded approximately 20 additional hours of complementary work.

The data collected in both measurements of the knowledge test have been processed in a descriptive way, which allowed the determination of a score that later served as a basis to advance a process of hypothesis testing for paired samples with small probability samples of 13 students who wished to participate in a counseling program against time that demanded approximately 20 additional hours of supplementary work.

3. Results and discussion

3.1. Preliminary Analysis

The transformation of the concept of derivative had an evolutionary process in which it was promoted by different associated concepts, initially with the problem of tracing tangents initiated in ancient Greece with geometric problems, and then the determination of maxima and minima, formally enunciated by Fermat, later Newton and Leibniz. They took infinitesimal calculus and, therefore, the concept of derivative to its threshold. Based on the above, a description of the various key stages for the formation of the concept of the derivative is presented from the works of Pino et al. (2011), Ortega and Sierra (1998) and Pinto and Parraguez (2015). Through Table 1, the traceability of the evolution of the concept of the derivative is organized.

Table 1. Epistemological origin of the concept of derivative

320 -260 BC	Greek mathematicians were the first to consider the first notions prior to the concept of derivative, with the tracing of tangent exposed in book III of Euclid's Elements; proposition XVII, which poses the situation of tracing a tangent line at a certain point of a curve (circle), since in Greek mathematics there was no standard geometric method for the tracing of tangents, which led to consider them in particular cases. This proposition involves the concepts of the circle, perpendicular lines, radius, line segment, equality of angles and tangent. Euclid states the tangent line to a circle as "if a line is a tangent to a circle, the radius is traced at the point of contact, this radius is perpendicular to the tangent" Vera (1970) cited in ( ). Also, in the books of Apollonius of Pergamon, a study of tangents and determination of maxima and minima in conic sessions can be visualized.
XIV Century	Towards the Middle Ages, variation studies were considered through geometric-descriptive sections, focused on the understanding of physical problems related to movement in general and particular is an object thanks to Merton College and Oresme, in the so-called Merton's rule that involves the graph of the distance traveled at a uniform acceleration. However, the procedures developed were mainly by representing particular cases, which were subjected to the analysis of situations represented geometrically and presenting argumentative or visual procedures since there were no methods of generating variation problems, developing premature intuitions of the notion of limit.
XVI Century	Oresme's work on motion led Galileo to study the vectorial composition of motion and to describe the trajectory of a projectile representing its motion from a graph where displacement and time are related, instructing ideas of tangent tracing to different curves. Thus, through kinematic argumentation, tangent concepts are reinforced regarding the composition of two movements. These methods of drawing tangents were generalized and extensive without considering the infinitesimal part, avoiding the determination of all velocities.
XVII Century	At this point, algebra is articulated to study problems related to geometry and equations of coordinate systems, leaving aside the geometric-descriptive study. Although Descartes, for example, posed geometric problems such as the tangent in algebraic expressions, as his method of geometric circles, the difficulty before the analysis and solution of problems from algebraic equations and analytical geometry is that it became tedious and extensive when expressing some curves in algebraic terms due to the large calculations. At this time, many mathematicians tried to give a general solution to the notion of a tangent. Fermat and Barrow developed methods for determining the tangent and maxima and minima, analyzing the infinitesimal proximity of the secant to the tangent, thus constructing

	the first intuitive ideas of limits, and derivatives from algebraic, geometric and infinitesimal sections. Newton and Leibniz are credited with the invention of infinitesimal calculus, with some considerations in their methods, on the part of Newton genus algorithms for the determination of maxima and minima, tangents and curvatures, making an understanding with infinitesimal sense, equal to that presented by Barrow and Fermat making a mechanical conceptualization, related to continuous motion, called configuration of fluxions, being the variables x and y the motion of the fluxions what is commonly known as the derivative. So in that sense, Leibniz presented an interest in developing better clarity of the concepts involved in the determination of maxima, minima, tangents and inflection points, presenting what is known as differential calculus and exposing some derivative formulas.
18th century onwards	From this time on, the foundations for the formalization of concepts related to infinitesimal calculus began to be laid, starting with the concepts of function, limits (being essential for the understanding of the derivative) and derivatives from a more formal perspective, with a predominantly algebraic character and sometimes articulating with the geometric, thanks to the different contributions of mathematicians such as Euler, Cauchy, Lagrange, among others.

Source: Saraza-Sosa, et al., 2019.

3.2. A Priori analysis

In the a priori analysis, the state-of-the-art type review has been taken into account in several antecedents that have focused on the determination of several difficulties or skills commonly presented by students and teachers when the conceptualization of the derivative of a function is oriented.

Table 2. Background reviews

Authors	Description
Sanchez-Matamoros et al. (2006)	Students show a comprehensive development of the concept of the derivative when different representative means are articulated, as in the case of graphical and analytical use, as opposed to when they are asked about situations in which the punctual and global character is related. $f'(a)$ and global $f'(x)$. In this sense, they recommend articulating the two in these situations.
Cervantes-Salazar et al. (2008)	They present a didactic proposal in teaching the derivative from a perspective focused on modeling context situations, with the objective of changing the traditional approach with abstract contents usually taught. They conclude that there is a better appreciation of the concept of derivative by the students, from the modeling approach to the traditional one.
Salazar et al. (2009)	They observe tendencies on the part of some students to consider the derivative as an algorithmic process without relating it to the rate of change or the slope of the tangent line; likewise, they can apply the criteria of the first and second derivative, but they do not relate it to the derivative function; also there is no relation between the graph of the function and the derivative function.
Ariza and Llinares (2009)	In the population studied, the concept of the derivative was analyzed in the articulation of economic situations, demonstrating that there is a strong inclination to use the algebraic register in comparison to the students' use of graphic language, sustaining that the students who used both languages had greater assimilation of the concept of the derivative.
Carabús (2009)	They applied a test in order to indicate the conceptions presented by the students regarding the subject of the derivative, arriving at various conclusions relating the derivative as the one that determines the growth and decrease in the function, a single value of the slope of the tangent line, as an algorithm applied to functions

	without meaning, among others. The authors generally note shortcomings in the conception of variational and graphical thinking.
Vrancken et al. (2010).	In the implementation of various means of representation, students show greater ease when working in numerical registers and difficulties in graphic and algebraic languages; in this sense, they conclude that the proposal presented, which enhances the variational approach, promoted a better understanding of the idea of the concept of derivative.
Badillo et al. (2011)	Describes the levels of understanding between the relationships of $f'(a)$ y $f'(x)$ that five mathematics teachers possess, determining variability in the graphical to algebraic assimilation of the respondents, exposing that some ambiguities in the relation $f'(a)$ and, difficulties in the relation of $f(x)$, $f'(x)$ y $f'(a)$ and the non-argumentation in the process of derivative determination through rule or the limit.
Orhun (2012)	He determined conceptual errors on the part of the students, arguing that an understanding of the subject of the derivative is not developed since they are related to operative actions without any graphical interpretation of the derivative of a function. The author also postulates difficulties in the connections between the original function and the derivative function, attributing to $f'(x)$ to be the graph.
Alvarez et al. (2013)	They analyze the orientation of the concept of derivative in two groups, one control group where a traditional teaching methodology was implemented and the other experimental group where knowledge was imparted through didactic strategies focused on strengthening previous concepts of symbolic, conceptual and procedural nature. It is concluded that even in both groups, ideas of the concept of the derivative are built, but the experimental group achieves significant conceptual and procedural learning in problem-solving.
Vega et al. (2014)	They identified the learning developed by students in the concept of derivative and its respective applications, finding weaknesses in the assimilation of this concept in terms of the derivative of a general and particular function at a given point and limitations in the assimilation from a visual and geometric perspective.
Vrancken and Engler (2014).	They implemented a learning sequence focused on promoting the conceptual assimilation of the derivative, structuring situations of variation in the context of physical situations, such as calculation of ratios of change and slopes, addressing various means of representation such as verbal, graphical, numerical and analytical, making relationships between them. The difficulties presented by the students were of conceptual and algorithmic type related to the necessary previous knowledge.
Hashemi et al. (2014).	They postulate results showing several difficulties in the learning of the derivative. First, from the instrument applied, they obtain deficiencies in conceptual questions; answers focused more on the symbolic aspect of the derivative than on the graphic understanding (limited geometrical interpretation of the concept), without a relationship between the two.
Cuevas et al. (2014)	They introduce the concept of the derivative of a real function through the use of digital resources, where they emphasize not using one means of representation but articulating different representative means. In the results, the students initially presented deficiencies in arithmetic, algebraic and function concepts. However, the intuitive development of the derivative of a real function was promoted from the use of technological tools.
Pino-Fan et al. (2015).	Applying an instrument to teachers in training, several difficulties were characterized in the solution of situations related to the concept of the derivative from different perspectives, such as the relation of the derivative with the instantaneous rate of change, the difference between f' (x) and f' (a). Better assimilation is exhibited when considering the derivative as the slope of the tangent line.
Santoyo et al. (2016)	Two groups were established, one with a traditional methodology and the other using an experimental approach, obtaining in the latter a better appreciation of the concept of the derivative of a function and the interpretation of different

	representation registers, thus developing skills in the resolution of optimization problems.
Domínguez-Contreras & Sanchéz-Galeano (2016).	In the design and application of an instrument to classify the geometric reasoning of the concept of the derivative, it was found that students made visual statements of the graphs without recognizing the geometric representation of the derivative, developed informal processes in the determination of the slope, from illustrations of the given slope and only a small population understood the process of finding the slope of the tangent line to the graph related to the derivative of a function.
Gutiérrez et al. (2017).	They characterize the difficulties presented by students about the derivative of real functions based on the analysis and surveys applied. Cognitive and procedural deficiencies are enunciated with the understanding of the derivative as a ratio of change, conceptualization of the derivative from the definition of limit, the derivative at a specific point, as well as algebraic and arithmetic difficulties in the application of the rules of derivation, thus proposing the implementation of a virtual learning object (OVA) focused on the learning of this knowledge.

3.3. Experimentation

Based on the epistemic analysis carried out where the key concepts in the development of the derivative were identified from its historical evolution, together with the a priori analysis of the different investigations in which the difficulties that commonly arise in the construction of this concept have been identified, a knowledge test was established focused on characterizing the conceptions presented by a group of first semester students of the academic program of Bachelor's Degree in Mathematics.

3.3.1. Knowledge test or Pre-test

Five situations were constructed to identify the students' deficiencies or potentialities in the different representation registers (algebraic, numerical and graphic) involved in determining different concepts related to the derivative of a function at a given point. Table 3 shows the characteristics of each of the items of this test.

Table 3. Description of the knowledge test

Item	Description	Type of record
	Four exercises were proposed in which they had to apply higher-order derivatives (second and third derivatives) from algebraic polynomials, fractions and expressions with radicals; in all cases, the chain rule had to be applied. The result obtained had to be simplified algebraically.	From algebraic to algebraic register
	The algebraic expression of a cubic function is presented to sketch the graph, draw and determine the slope of the tangent line at two given points and finish with the graphical representation of the derivative function.	From algebraic to graphic
	The algebraic expression of a complete cubic function is proposed to sketch the graph, starting from the first and second derivative criteria to identify intervals of growth, concavity and inflection points.	From algebraic to graphic

Three situations are proposed in a context where the function that must be proposed by the students from the data given by the statement is solved by optimizing the function. They must choose only one of the three proposed situations and solve them.	From everyday language to algebraic language
The graph of an upwardly concave quadratic function whose vertex has vertical displacement two units down and horizontal displacement three units to the left is given. The coordinates of the vertex are the key to obtaining the function's equation and its respective derivative.	From graphic to algebraic and graphical.

Table 4 shows the level of performance demonstrated by each student in the knowledge test on both qualitative and quantitative scales. In order to report an analysis of what was done, the following rubric is used to define the rating scale, considering that each item adds one point: a) poor performance equivalent to a rating of 0.25, which is assigned when the student reviews the statement of the proposed situation but what is done is not coherent with what is required; b) low performance equivalent to a rating of 0.50, characterized by the completion of processes but with the presence of errors; c) poor performance equivalent to a rating of 0.25, which is assigned when the student reviews the statement of the proposed situation but what is done is not coherent with what is required; d) low performance equivalent to a rating of 0.50 characterized by the realization of processes but with the presence of errors; c) medium performance equivalent to a valuation of 0.75 that is adjusted to a correct reading of the proposed situation, to the correct design and application of a solution method but that does not contextualize the result found; d) high performance corresponding to a valuation of 1.00 characterized by the correct resolution of the proposed situation accompanied by a process of reasoning and contextualization of the answer.

Table 4. Pre-test results

Informant	tem 1	Item 2	tem 3	Item 4	tem 5	Rating
E1	nder	Under	oor	Poor	oor	1.75
E2	igh	Under	oor	Poor	oor	1.50
E3	igh	Medium	oor	Poor	oor	1.75
E4	nder	Under	oor	Poor	oor	1.75
E5	nder	Under	oor	Poor	oor	1.50
E6	nder	Under	oor	Poor	oor	1.25
E7	nder	Medium	oor	Poor	oor	2.00
E8	nder	Medium	oor	Poor	oor	1.25
E9	nder	Medium	oor	Poor	oor	1.25
E10	nder	Under	oor	Poor	oor	1.75

	E		U		P	1.75
11	nder	nder	oor	oor	oor	
	E		U		P	2.00
12	edium	nder	oor	oor	oor	
	E		U		P	1.75
13	nder	nder	oor	oor	oor	

The following is a report of what the students did or proposed for each item.

- *Item 1.* It was oriented to identify the existence of algebraic thinking in the algorithmic solution of the exercises presented, which required the application of different derivation rules such as the product rule, the quotient rule, and the chain rule, among others, to obtain higher-order derivatives of the functions presented. The polynomial function was determined not to generate any difficulty in the students. However, in the other expressions in which the level of complexity was increasing, it was possible to evidence deficiencies both in the handling of the derivative of the quotient and in the application of the chain rule, which corresponds to the basic knowledge of the course. Additionally, the solution process was complicated because errors were presented around basic topics such as the handling of exponents and arithmetic operations with fractions.
- *Item 2.* It was proposed to analyze the relationship in the graphic register between the function f(x) and the derivative function f' (x), distinguishing the notions regarding the tangent line at any point and at different points with respect to the given function. With this, it was expected to determine if the comprehension of the graphic register of the derivative is greater than the algebraic register, which was analyzed in the previous item. The students resorted to the tabular register to generate no more than six points associated with the function to propose a graphical representation of the function in the plane. A wrong concept is observed around the tangent line to a curve at a point since they represented were secant lines at each given value of the abscissa. They calculate the derivative algebraically but do not know how to interpret it or its relation with the slope of the tangent line, so they could not obtain the equation of the tangent line. Then, from the scarce argumentation of the processes carried out, it can be inferred that there is poor knowledge of the function and its derivative from the graphic register.
- *Item 3. This* item seeks to identify the student's level of understanding of the first and second derivative criteria to sketch the graph of a function from the identification of its various characteristics, for which they were provided with a cubic function with three elements. From what was proposed by the students, it could be observed that they limited themselves to calculating the first and second derivatives algebraically, but from there, they did not advance any further. Therefore, they could not determine the growth intervals, maximum or minimum values, inflection point or concavity intervals, much less sketch the graph. Finally, two students sketched the graph using a table of values.
- *Item 4.* Students were expected to interpret the first derivative as a criterion to identify the presence of maximum or minimum values within a situation in context. Three different contexts were proposed, one arithmetic and one geometric, where students had to construct

the function from the data given in the statement. The function was provided as a complementary element in the last situation, which was associated with variational thinking. This was done in order to identify the students' preferences for situations. As a general conclusion, 90% of the students were inclined to the third situation where the algebraic expression of the function was given, but they continued replicating the same process; that is, they calculated the derivative of the function algebraically, but from there they do not advance in the interpretation of anything about it.

- *Item 5.* This item is proposed to determine the function corresponding to a graphical representation in the Cartesian plane. They had to apply the concepts of vertical and horizontal displacement of the vertex of a quadratic function to construct its algebraic expression and, with this data, advance to the calculation of the derivative algebraically and then represent it in the Cartesian plane together with the original function. All the students proposed a diversity of possible processes to determine the algebraic expression associated with the graph. However, all of them were incorrect, evidencing the lack of knowledge of the basic principles of graphing that are part of the mathematical graphical record.

3.3.2. Pedagogical Intervention

From the results obtained in the knowledge test, it was possible to identify that there was a long list of difficulties surrounding the concept of the derivative, which became more acute when students had to solve situations in context or articulate different semiotic registers of representation that started from the traditional algebraic register.

These findings made it possible to organize a pedagogical sequence developed in eight class sessions of two hours each, emphasizing the understanding of concepts and processes from articulating diverse semiotic registers of representation. Furthermore, the pedagogical work was carried out in teams of no more than three students. The students' propositional work and reasoning were always privileged, with a teacher who only accompanied and guided the actions without giving solutions but always encouraged creativity and discussion of possible alternative solutions.

3.3.3. Knowledge Test or Posttest

Once the pedagogical process of the development of the didactic sequence with the students was finished, immediately after the next class, the students presented the knowledge test again without prior notice of its application through Table 5.

Table 5. Post-test results

Info rmant	tem 1	tem 2	I tem 3	It em 4	tem 5	Ratin g
E_1	nder	nder	U nder	U nder	edium	2.75
E_2	igh	igh	M edium	M edium	igh	4.50
E_3	igh	igh	P oor	M edium	igh	4.00
E_4	edium	edium	M edium	M edium	igh	4.00

E_5	High	High	Medium	Medium	High	4.50
E_6	High	High	Medium	Medium	High	4.50
E_7	High	High	High	Medium	High	4.75
E_8	High	High	High	Medium	High	4.75
E_9	High	High	High	Medium	High	4.75
E_10	High	High	High	Medium	High	4.75
E_11	High	High	High	Medium	High	4.75
E_12	High	High	High	Medium	High	4.75
E_13	Under	Under	Medium	Medium	Under	3.00

4. Posteriori Analysis

This corresponds to the last stage of the research process and, given the nature of didactic engineering, it is intended to validate the effectiveness of the pedagogical intervention in the light of the so-called internal validation, which corresponds to the comparison of the results obtained in both measurements, that is, the comparison of the results between the pre-test and the post-test. When comparing the assessment criterion in the totality of the items reported for the members of the sample, an improvement in the performance levels can be evidenced, which can be reflected in Figure 1.

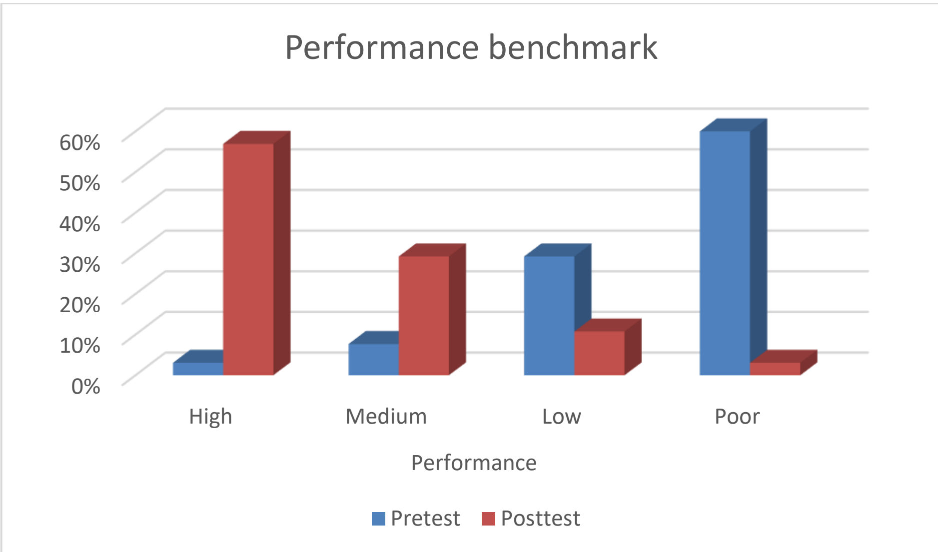

Figure 1. Comparative performance by test.

In order to statistically validate the improvement in the performance obtained in the test, the application of the hypothesis test for paired samples was used for the difference in means, which in this case corresponds to the difference in the scores obtained between the two moments of measurement (Posttest and Pretest). As a result, the following hypothesis system was validated:

Null hypothesis H_o $$diferencia_{promedio}(Postest - Pretest) \leq 0$$

Alternative hypothesis H_a $$diferencia_{promedio}(Postest - Pretest) > 0$$

Table 6 shows the calculations of the differences in the ratings for each member of the sample.

Table 6. Difference in qualifications

Informant	re-test	Post-test	Difference $(Pos - Pre)$	$Diferencia^2$
E1	.75	1.75	1.00	1.00
E2	.50	1.50	3.00	9.00
E3	.00	1.75	2.25	5.06
E4	.00	1.75	2.25	5.06
E5	.50	1.50	3.00	9.00
E6	.50	1.25	3.25	10.56
E7	.75	2.00	2.75	7.56
E8	.75	1.25	3.50	12.25
E9	.75	1.25	3.50	12.25
E10	.75	1.75	3.00	9.00
E11	.75	1.75	3.00	9.00
E12	.75	2.00	2.75	7.56
E13	.00	1.75	1.25	1.56

The average difference, the standard deviation of the difference and the test statistic are calculated through equations 1, 2 and 3,

$$dif_{mean} = \Sigma\, dif / n = 34.50 / 13 = 2.65 \qquad \text{Equation 1}$$

$$S_{dif} = \sqrt{\frac{\sum dif^2 - n(dif_{mean})^2}{n-1}} = \sqrt{\frac{98.88 - 13(2.65)^2}{13-1}} = 0.78 \qquad \text{Equation 2}$$

$$t = \frac{dif_{mean}}{\frac{S_{dif}}{\sqrt{n}}} = \frac{2.65}{\frac{0.78}{\sqrt{13}}} = 12.25 \quad \text{Equation 3}$$

obtaining a test statistic value of 12.25, with a 95% confidence level and significance level with an alpha of α = 5%≈0.05. Given the null hypothesis statement, the hypothesis test has that alpha value concentrated in the right tail. Because the sample size is less than 30, it is assumed that the test statistic is obtained from the T-Student Distribution table, locating the significance level associated with a sample size of n-1 = 12 in order to determine the critical value that determines the acceptance and rejection zones, which corresponds to the value of t_value= 1.7823.

Given that the test statistic calculated with the students' data was t_calculated=12.25 is greater than the test statistic t_value= 1.7823, the conclusion is reached that it falls in the rejection zone, which is equivalent to affirming that, based on the information obtained from the sample, there is insufficient evidence to accept the null hypothesis, that is, to accept that after the application of the pedagogical intervention, better results were obtained in the mathematical knowledge test, with a confidence level of 95%.

5. Conclusions

Based on what was found in the epistemic analysis, it is evident that the concept of derivative had an evolution from a geometric and visual perspective until its respective formalization in algebraic terms, contrary to the findings found in the documentary type tracking in which the students develop a strong notion of the derivative of a function as the algorithmic procedure in the application of the different rules, despising to a great extent the respective graphic understanding. The above is similar to what was found in the knowledge test conducted in this research, where it was also possible to identify shortcomings in resolving the points presented. Thus, in the application of the activities that articulated the graphic and algebraic registers, the lack of conceptual knowledge that would allow them to comply with what was requested is evident; an example is the poor tracing that some students gave to the tangent line in the given points and, likewise, the arguments of the notions they possessed.

The situations that had the purpose of identifying how they use the derivative in terms of application to their respective concept in the determination of critical points or optimization problems were those that generated less acceptance, or development close to a correct solution, leaving only the parts that were developed mechanically, without proposing anything else; a situation that was solved in the post-test where the students evidenced better performances from work carried out in the pedagogical intervention. The above implies that part of the students' deficiencies is associated with the pedagogical process that the teacher privileges in the classroom since when approaching the concepts articulated with the different semiotic registers of representation, the informants evidenced better performances.

Based on the above, it is recommended to teach this concept with the articulation of different registers of semiotic representation in order to form its assimilation from different perspectives;

this premise leads to the permanent improvement of the teaching work towards the offering of better teaching practices that contribute to the understanding of mathematical concepts.

References

Álvarez, D., Colorado, H. & Ospina, L. P. (2013). Una propuesta didáctica para la enseñanza del concepto de derivada. *Revista Científica,* 104-110.

Ariza, Á. & Llinares, S. (2009). Sobre la aplicación y uso del concepto de derivada en el estudio de conceptos económicos en estudiantes de Bachillerato y Universidad. *Enseñanza de las Ciencias, 27*(1), 121-136.

Artigue, M., Douday, R., Moreno, L., & Gómez, P. (1995). *Ingeniería Didáctica en Educación Matemática. Un esquema para la investigación y la innovación en la enseñanza y el aprendizaje de las matemáticas*. México: Grupo Editorial Iberoamérica.

Badillo, E. R., Azcárate, C. & Font, V. (2011). Análisis de los niveles de comprensión de los objetos f'(a) y f'(x) en profesores de matemáticas. *Enseñanza de las Ciencias, 29*(2), 191-206.

Briceño, E., Hernández, J. & Espino, A. (2018). Análisis de la comprensión de la derivada desde el enfoque gráfico en estudiantes de nivel superior. *El Cálculo y su Enseñanza, Enseñanza de las Ciencias y la Matemática, 10,* 31-47.

Calderón, D. I. & León, O. L. (2012). La ingeniería didáctica como metodología de investigación del discurso en el aula. En D. Calderón & O. L. León, *Lenguaje y Educación: Perspectivas metodológicas y teóricas para su estudio* (pp. 71-104). Colombia: Universidad Distrital Francisco José de Caldas.

Carabús, O. (2009). El Aprendizaje del Cálculo en la Universidad. La Conceptualización de la Derivada de una Función y sus Niveles de Comprensión. En *Congreso regional de ciencia y tecnología* (pp. 1-11). Catamarca: Universidad Nacional de Catamarca.

Cervantes-Salazar, M., Camarena-Gallardo, P. & Pinet-Plasencia, R. (2008). La derivada con la matemática en contexto y el enfoque hacia la modelación. *Científica, 12*(4), 167-173.

Cuevas, C. A., Rodríguez, A., & González, O. (2014). Introducción al concepto de derivada de una función real con apoyo de las tecnologías digitales. En P. Lestón (Ed.), *Acta Latinoamericana de Matemática Educativa* (pp. 2335-2345). México, DF: Comité Latinoamericano de Matemática Educativa.

Domínguez-Contreras, V. & Sanchéz-Galeano, M. (2016). El concepto de derivada en estudiantes de educación media. *Eco Matemático, 7*(1), 86-91. DOI: https://doi.org/10.22463/17948231.1105

Duval, R. (2006). Un tema crucial en la educación matemática: La habilidad para cambiar el registro de representación. *La Gaceta de la Real Sociedad Matemática Española, 9*(1), 143-168.

Duarte, L. C., Ricardo, N., & Santos, L. V. (2018). Dominio afectivo de los estudiantes de educación media hacia las matemáticas. *Revista Perspectivas, 3*(2), 60–71. DOI: https://doi.org/10.22463/25909215.1589

Gallo, A., Manrique, J., & Prada, R., (2017). Aplicación de la ingeniería didáctica en el aprendizaje del concepto de función. *Eco matemático 8*(1). 43-48

García, J. Á. (2013). La problemática de la enseñanza y el aprendizaje del cálculo para ingeniería. *Revista Educación, 37*(1), 29-42.

Gómez, J. A. (2020a). *Transformaciones curriculares en la educación superior en América Latina*. Bogotá: Editorial FEDICOR.

Gómez, J. A. (2020b). *Sobre la calidad en la educación superior: aportes desde la revisión documental y bibliográfica en Colombia y América Latina*. Tunja: Fundación Universitaria Juan N. Corpas. Centro Editorial. Ediciones FEDICOR

Gutiérrez, L., Buitrago, M. R. & Ariza, L. M. (2017). Identificación de dificultades en el aprendizaje del concepto de la derivada y diseño de un OVA como mediación pedagógica. *Revista Científica General José María Córdova, 15*(20), 137-153. DOI: https://doi.org/10.21830/19006586.170

Hashemi, N., Abu, M. S., Kashefi, H. & Rahimi, K. (2014). Undergraduate students' difficulties in conceptual understanding of derivation. *Procedia-Social and Behavioral Sciences*, *143*, 358-366.

Hernández-Suarez, C. A., Prada-Núñez, R., & Ramírez-Leal, P. (2017). Obstáculos epistemológicos sobre los conceptos de límite y continuidad en cursos de cálculo diferencial en programas de ingeniería. *Revista Perspectivas, 2*(2), 73-83. DOI: https://doi.org/10.22463/25909215.1316

Hitt, F. (2003). *Dificultades en el aprendizaje del cálculo*. www.academia.edu/807014/Dificultades_en_el_aprendizaje_del_cálculo

Orhun, N. (2012). Graphical understanding in mathematics education: Derivative functions and students' difficulties. *Procedia-Social and Behavioral Sciences*, *55*, 679-684.

Ortega, T. & Sierra, M. (1998). El concepto de derivada: algunas indicaciones para su enseñanza. *Revista Interuniversitaria de Formación del Profesorado,* (32), 87-115.

Park, J. (2013). Is the derivative a function? If so, how do students talk about it?. *International Journal of Mathematical Education in Science and Technology, 44*(5), 624–640. DOI: https://doi.org/10.1080/0020739x.2013.795248

Pineda, W. B., Hernández, C. A., & Rincón, O. L. (2019). Estrategias para la enseñanza de la matemática: una mirada desde los docentes en formación. *Revista Perspectivas, 4*(1), 48–53. DOI: https://doi.org/10.22463/25909215.1759

Pino, L. R., Godino, J. D. & Moll, V. F. (2011). Faceta epistémica del conocimiento didáctico-matemático sobre la derivada. *Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática, 13*(1), 141-178.

Pino-Fan, L. R., Godino, J. D. & Font, V. (2015). Una propuesta para el análisis de las prácticas matemáticas de futuros profesores sobre derivadas. *Bolema-Boletim de Educação Matemática, 29*(51), 60-89.

Pinto, I., & Parraguez, M. (2015). El concepto de derivada desde la teoría los modos de pensamiento, sustentada en la epistemología de Cauchy. En R. Flores (Ed.), *Acta Latinoamericana de Matemática Educativa* (pp. 337-344). México, DF: Comité Latinoamericano de Matemática Educativa.

Prada, R., Hernández, C.A., & Jaimes, L.A. (2017). Representaciones semióticas alrededor del concepto de función en estudiantes de ingeniería. *Góndola, enseñanza y aprendizaje de las ciencias, 12*(2), 14–31. DOI: https://doi.org/10.14483/23464712.10491

Prada-Núñez, R., Hernández-Suárez, C. A., & Ramírez-Leal, P. (2016). Comprensión de la noción de función y la articulación de los registros semióticos que la representan entre estudiantes que ingresan a un programa de Ingeniería. *Revista Científica, 25*(2), 188–205. DOI: https://doi.org/10.14483//udistrital.jour.RC.2016.25.a3

Ramírez, L, Gómez, J, Batanelo, L, Chica, F, Marin, J., & Tuay, R. (2020). *La calidad de la educación en el laberinto del siglo XXI: aportes desde el proyecto de investigación sobre incidencia del enfoque pedagógico en la calidad de la educación.* Tunja: Fundación Universitaria Juan N. Corpas. Centro Editorial. Ediciones FEDICOR.

Rodríguez-Pérez, E. (2015). El concepto de derivada y el modelo de Van Hiele en estudiantes de licenciatura en matemáticas e informática de la Universidad. *Eco Matemático, 6*(1), 43–49. DOI: https://doi.org/10.22463/17948231.458

Rojas, R. R., & Del Rosario, E. (2020). Aplicación de la resolución de problemas de Pólya en el estudio de ángulos en estudiantes de cuarto grado del nivel secundario. *Revista Perspectivas, 5*(2), 6–12. DOI: https://doi.org/10.22463/25909215.2823

Ruiz, E. F., Gutiérrez, J. J. & Garay, L. I. (2018). Visualizando problemas de la derivada con aplicaciones en dispositivos móviles. *Innovación educativa, 18*(76), 39-67.

Salazar. C., Díaz, H. & Bautista, M. (2009). Descripción de niveles de comprensión del concepto derivada. *Tecné Episteme y Didaxis TED,* (26), 62-82.

Sánchez-Matamoros, G., García, M., & Llinares, S. (2006). El desarrollo del esquema de derivada. *Enseñanza de las Ciencias, 24*(1), 85-98.

Santoyo, F., Rangel, M. Á., Santoyo, E. & Puga, K. L. (2016). Evaluación de una estrategia didáctica para la apropiación del concepto “derivada de una función”. *RIDE. Revista Iberoamericana para la Investigación y el Desarrollo Educativo, 7*(13), 250-272.

Saraza, D., Prada-Núñez, R. (2017). Estado del arte alrededor de la comprensión conceptual de la derivada. En R: Prada-Núñez, P. Ramírez, C. Hernández, H. Gallardo, S. Mendoza, G. Rincón (eds.), *Encuentro Internacional en Educación Matemática* (pp. 122-128). Cúcuta: Universidad Francisco de Paula Santander.

Saraza-Sosa, D. F., Prada-Núñez, R., & Vera-Gutiérrez, C. L. (2019). Comprensión de la derivada apoyada en la ingeniería didáctica como método de investigación. *Eco Matemático, 10*(1), 42–56. DOI: https://doi.org/10.22463/17948231.2542

Tall, D. (1993). Students’ difficulties in calculus. *Proceedings of Working Group 3 on Students’ Difficulties in Calculus, ICME-7*, 13-28.

Trujillo, J. A., Vera, C. L., & Saraza, D. F. (2019). Ingeniería didáctica como recurso metodológico para el aprendizaje de los conceptos de límite y continuidad. *Revista Perspectivas, 4*(1), 39–47. DOI: https://doi.org/10.22463/25909215.1758

Vega, M. A., Carrillo, J. & Soto, J. (2014). Análisis según el modelo cognitivo APOS del aprendizaje construido del concepto de la derivada. *Boletim de Educação Matemática, 28*(48), 403-429.

Vrancken, S. & Engler, A. (2014). Una Introducción a la Derivada desde la Variación y el Cambio: resultados de una investigación con estudiantes de primer año de la universidad. *Boletim de Educação Matemática, 28*(48), 449-468.

Vrancken, S., Engler, A. & Müller, D. (2010). Una secuencia didáctica para la introducción del concepto de derivada. Resultados de su implementación. En P. Lestón (ed.), *Acta Latinoamericana de Matemática Educativa* (pp. 379-388). México, DF: Comité Latinoamericano de Matemática Educativa A. C.

Weigand, H. G. (2014). A discrete approach to the concept of derivative. *ZDM*, *46*(4), 603-619.