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Exact And Approximate Sequential Methods In Solving The Quadratic Assignment Problem

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Abstract

This article provides a description of the most common methods in solving the quadratic assignment problem with extensive reference material. This problem is a difficult combinatorial problem whose solution (for any instance) is even impossible to establish within a given radius unless the class of problems P coincide with the class of problems NP. The QAP is considered one of the most complex combinatorial optimization problems and is the model for many real life problems such as facility layout, campus planning, backboard wiring, scheduling, computer manufacturing, turbine balancing and process communications among other applications.
Another reason that highlights the structure and complexity of QAP is that many others significant NP-Complete combinatorial optimization problems result in particular cases of QAP, some of them are: The Traveling Salesman Problem (TSP): How an agent should visit a set of n cities returning to the starting city in such a way that each city is just visited once and the cost of the tour is the minimum?. Maximum Clique Problem (MCP): Given a graph $G = (V, E)$ with $|V| = E$, this problem consists in finding the maximum number $k \leq n$ such that there exists a subset V_1 de V with k vertices which induces a clique in G

Keywords: Quadratic assignment problem, combinatorial problem.

Introduction

The Quadratic Assignment Problem consists of assigning a set of *n* facilities in a set of *n* locations, knowing the flow between facilities and the distances between locations. The goal is to assign each facility to each location in such way flow and distances are minimized.

The mathematical model is:

$$\text{Min}_{\sigma \in S_n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f_{ij} d_{\sigma(i)\sigma(j)}$$

where $D = (d_{ij})$ is a distance matrix, $F = (f_{ij})$ is a flow matrix, $S_n = \{\sigma \mid \sigma: N \to N\}$, where $N = \{0,1, \dots, n-1\}$.

The QAP was introduced by Koopmans & Beckmann in 1957 [1] and was mathematically proven to belong to the NP-Complete Problems category by Sahni & Gonzalez in 1976 [2].

The QAP is considered one of the most complex combinatorial optimization problems and is the model for many real life problems such as facility layout, campus planning, backboard wiring, scheduling, computer manufacturing, turbine balancing and process communications among other applications [3].

Some exact methods, like Branch and Bound, Cutting Plane and Bender's decomposition [4, 5, 6, 7, 8] have been applied to solve instances of this problem. However, the amount of resources required by these methods make them non-applicable to instances of size *n > 30* [9, 10]. In order to tackle this issue, some approximation methods, like Greedy Randomized Adaptive Search, Tabu Search, Simulated Annealing, ACO (Ant Colony Optimization), PSO (Particle Swarm Optimization), and Evolutionary Algorithms (especially Genetic Algorithms)
[11, 12, 13, 14, 15, 16, 17, 18, 19] have been applied to find near optimal solutions to instances of the QAP

Techniques to solve the QAP

A summary of the state of the art of sequential techniques (exact methods and approach methods) to solve the QAP are:

Exact Methods

Correspond to mathematical programming techniques, some of them are based on the divide and conquer strategy, i.e. partition the search space of the problem in sub-problems and to optimize each one of them separately. the most common are:

- **Branch and bound (B&B).** It is based on an implicit enumeration of all problem's solutions. The search is carried out on the whole problem domain; the set solutions is thought of as forming a rooted tree, the root represents the problem itself, and the leaves additional restrictions that bound the solution to the problem.
  The solution of the problem improves as the iterations advance. The branching operator that determines the order in which the branches are explored is used (for example, a deep search or a wide search) and the pruning operator that eliminates solutions that do not take to the best, considering lower bounds for every partial solution. The method ends when there are no more branches or nodes to explore or when all the nodes have been eliminated. The quality of the bounds and the branching strategy determine the quality of the method. Generally, the QAP is solved by B&B from its representation as linear assignment problem [20] and using bounding techniques, as Gilmore-Lawler lower bounds method [4].

- **Dynamic programming.** It is based on Richard Bellman's optimality principle “All sub-policy of an optimal policy must be optimal as well". Solving general problem in a recursive manner positing less complex problems that have the same structure of the original problems; this process is applied until problems with immediate solution are found, then, the process starts again until a solving general problem is found.

- **Relaxations.** A problem is solved with less requirements than the original problem, the most common relaxations are the linear programming relaxations and Lagrangian relaxations. The former solves a problem with real variables when the original requires only integer solutions, with this, lower bounds are found and they can be used with the B&B method for branching part. In the Lagrangian relaxation some restrictions of the problem (usually the most difficult) are removed and incorporated to objective function through a penalizing function [21]. Some relaxations for QAP appear in [5, 22, 23].

- **Benders decomposition.** QAP formulated as a linear programming (mixed integer) problem can be solved fixing integer variables and solving the corresponding dual problem. The time for this convergence to happen is long and it is usually applied only to small instances, but when applying cutting plans, good sub-optimal solutions are produced, [6].

- **Cutting planes.** Cutting planes were proposed by Gomory [24]. They are based on adding specific restrictions to relaxed Linear Programming problem. The method approximates the polyhedron represented by the convex shell of all feasible solutions of the original problem by the polytope of the relaxed problem.

Approach methods

They are divided in metaheuristics based of trajectory and metaheuristics based on population.

Metaheuristics based on trajectories. Consist of making determined searches in solutions space; they start with only one initial solution, solution in each iteration is replaced by another (frequently a better one). These methods have a spirit oriented to exploit promising regions of the search space (intensify the search). The most common in the solution of the QAP are:

- **Construction methods.** They are considered the simplest heuristics for the QAP. The quality of the solutions is not the best, however, they are very simple to implement computationally and given their properties they can be used as part of more intelligent methods for the QAP. Basically they start with an empty permutation, and recursively assign places to facilities according to some certain criteria until all facilities have been assigned. These methods were formulated for the first time by Gilmore [4] towards 1960. Another method of construction with better results is the one proposed by Muller in [25].

- **Local Search method (LS).** They are algorithms that produce optimal local solutions in the following way: a neighborhood $N(\sigma_0)$ of a permutation σ_0 of the QAP consists of all permutations that in some sense are close to σ_0, therefore a local optimal QAP solution is a permutation $\bar{\sigma}$ so that:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f_{ij} d_{\bar{\sigma}(i)\bar{\sigma}(j)} = \mathrm{Min}_{\sigma \in N(\sigma_0)} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f_{ij} d_{\sigma(i)\sigma(j)}$$

The procedure starts with an initial solution and is improved through a movement to a solution in its neighborhood; this procedure is repeated until no solution gets

better. In order to obtain better results, the local search algorithms are carried out many times starting at different initial solutions (It is clear to use a parallel implementation). A more complete study appears in [26].

- **2-opt heuristic.** The 2-opt local optimization heuristic is applied in order to improve candidate QAP solutions [50]. This method consists of performing all pairwise exchanges of all possible facilities on each of the locations in a particular permutation. A solution is updated with a better one as long as $\Delta_{ij} < 0$ in the following formula:

$$\Delta_{ij} = (f_{ij} - f_{ji})(d_{\sigma(i)\sigma(j)} - d_{\sigma(j)\sigma(i)}) + \sum_{\substack{k=0 \\ k \neq i,j}}^{n-1} ((f_{jk} - f_{ik})(d_{\sigma(i)\sigma(k)} - d_{\sigma(j)\sigma(k)})) + (f_{kj} - f_{ki})(d_{\sigma(k)\sigma(i)} - d_{\sigma(k)\sigma(j)})$$

with asymmetric distance and ow matrices [14], (*i, j* are facilities that are exchanged).

A matrix formulation for Δ_{ij} is:

$$\Delta_{ij} = (f_{ij} - f_{ji})(d_{\sigma(i)\sigma(j)} - d_{\sigma(j)\sigma(i)}) + (F_{i\cdot} - F_{j\cdot}) \cdot ((DX^t)_{\sigma(j)\cdot} - (DX^t)_{\sigma(i)\cdot}) + (F_{\cdot i} - F_{\cdot j}) \cdot ((XD)_{\cdot\sigma(j)} - (XD)_{\cdot\sigma(i)})$$

where the symbol · interprets an internal product, (f_{ij} = (f_{ji} = 0 in the addends 2 and 3. $F_{i\cdot}$_ indicates the row *k* of the matrix *F* and $F_{\cdot k}$ indicates the column *k* of the matrix *F*.

- **Greedy Randomized Adaptive Search Procedure (GRASP).** It is a very usual heuristic for combinatorial optimization problems. GRASP is a combination of greedy elements and randomized search elements, it is composed of a construction phase, where two facilities are assigned to two locations among all those with minimal costs and an improvement phase which includes randomized elements to avoid falling into local optima. In [12] a detail implementation for QAP is described.

- **Tabu Search (TS).** This technique is used to “remember" which solutions have already been visited and to abandon neighborhoods that have optimal locations.
In QAP the movements used are usually swaps but are controlled by a tabu list, which does not allow certain movements on the current solution, as movements change, the list is actualized during the search. The solution starts with an initial feasible solution, only if selected solution is not in the tabu list, the initial solution is updated by the selected solution (this new solution is not necessarily better than the initial) and the search in the neighborhood is repeated. Different studies relate the convenient size of the tabu list, with respect to the QAP, in [13] there is a deep study. The tabu search algorithm has parallel nature in its implementation dividing its search load between various processors. Parallel implementations are proposed in [27, 28].

- **Simulated annealing (SA).** It is an approach that exploits the analogy between combinatorial optimization problems and mechanical statistics problems (many particles physical system). The feasible solutions of a combinatorial optimization problem correspond to states of physical system and values of objective function correspond to the energy of physical system state. A material is heated and then slowly cooled to change its physical properties. the

heat causes that the atoms augment their energy and move from the initial positions that would correspond to local minimums in an optimization problem. The slow cooling produces low energy states (thermic balance) that would correspond to global minimums in optimization problem. Wilhelm et al. [29] obtain good results for QAP with a sophisticated simulated annealing approach.

Metaheuristics based on populations. They are iterative techniques that apply stochastic operators on a set of individuals (population), each individual corresponds to a coded version of a possible problem solution. The performance (quality) of each individual is evaluated through an aptitude function; variation operators on some of the individuals guide the whole population to high quality solutions, which confer to these metaheuristics a good exploration power. The most common metaheuristics applied to the QAP are the following:

- **Genetic algorithms (GAs)**. Genetic algorithms (GAs) is one of the most outstanding approaches in the field of evolutionary algorithms and are defined as general purpose iterative adaptive search procedures. This is the metaheuristic considered in this work. The GAs have the advantage that describe in an abstract and rigorous way the collective adaptation of population of individuals to given environment, based on behavior similar to natural system.
  A simple Genetic Algorithm usually begins with a population of individuals randomly generated or sometimes pre-established by previous experiences or generated by some heuristic procedure. The simple GA maintains the population of constant size and works iteratively as follows:
  During each step in the iteration (called generation) individuals are evaluated and assigned a fitness value. To form a new population (from the previous one) the selection operator is applied, which consists of choosing individuals with a probability proportional to their relative aptitude; this ensures that the expected number of times an individual is chosen is proportional to their relative performance in the population. It is expected that individuals above the average have more copies in the new population (higher probability of reproduction), while individuals below the average have more risk of disappearing. This operator acts as generator of “intermediate parents" who will be responsible for giving rise to a better next population than the previous one. To incorporate new individuals into the population, some genetic operators are required, such as the crossover operator (which simulates sexual reproduction) and/or the mutation operator (which simulates asexual reproduction). The crossover, which is the most important operator of recombination, consists of taking two individuals called parents (extracted by the selection operator) and generating two new individuals called offsprings, exchanging parts of the parents; sub-strings of parent chains are exchanged from a certain crossing point chosen randomly. With the crossover, the search is guided towards good regions in the problem domain. The mutation operator essentially offers the possibility of avoiding a premature convergence to a local optimum, changing each symbol in some chains with a reduced probability. In more sophisticated genetic algorithms, crossover and especially mutation do not necessarily have to remain constant throughout the simple AG process. In general, the crossover aims to combine the most characteristic features of parent chains and therefore increase the fitness of new individuals. The mutation only affects one individual at a time and its intention is to avoid a premature convergence to a local optimum. Without the mutation operator, potentially useful genetic material could be lost. Since Algorithms are stochastic procedures, their performance varies from one execution to another (unless the same random number generator is used); due to this, the average performance of several executions is more reliable and therefore more used than the results generated by a single execution of algorithm.

Genetic algorithms, like most stochastic iterative algorithms, do not guarantee convergence. They end after a maximum number of iterations or when a satisfactory solution is reached. It may also happen that the quality of the results is not improved in the iterative process, therefore, the algorithm must be stopped before reaching the maximum number of iterations.
A great number of research projects have been proposed for the QAP, but generally this method works better considering a hybrid approach with a Local Search algorithm or a Tabu Search [30, 31].

- **Ant Colony Optimization (ACO).** This heuristic imitates the behavior of ants when searching for food. The analogy between the performance of ants and the
solution of a combinatorial optimization problem resides in the following factors: the search space of ants corresponds to the set of feasible solutions for the optimization problem, each source of food corresponds to the value of the objective function. The adaptive memory component of each ant is the track of pheromones that accumulate in the less traveled roads. With respect to the QAP, the pheromone track of ACO, is the measure of attraction to locate a facility *i* in a location *j*, in [16] this procedure is described in detail. Other studies are gathered in [32]; Tseng and Liang, in [17] apply first a GA to find an initial population. A parallel ACO procedure for the QAP appear in [33].

- **Particle swarm optimization (PSO).** In this method a number of particles move through the search space with the objective of finding an optimal position (good solution). The particles communicate between themselves and the one with the best position (measured according to an aptitude function) mark influence in the rest. The particles adjust their movements systematically (position and speed) according to their own experience and according to the experience the rest of the swarm. This method is inspired in the social behavior of organisms as bird flocks or fish banks. Although in principle it is a method for continuous search spaces, it has also been applied to discrete optimization problems like the QAP (see [34, 35]) and to the TSP.

Conclusions

Solving the QAP is not an easy task. Many of these models are an important technique to find optimal or near optimal solutions for significant instances of the QAP. Using parallel models, such as, parallel genetic model (the distributed model (islands model), the cellular model (grid model)) and combining it with what has already been mentioned will surely improve the results obtained, and perhaps for this it is convenient to configure a cluster of GPUs or combine procedures in multicore architectures.

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